THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE III, 2000 ISSN 1221-454X
ELECTROTECHNICS, ELECTRONICS, AUTOMATIC CONTROL, INFORMATICS

SOME RESULTS ON THE MODELLING OF TSS MANUFACTURING LINES

Viorel MÎNZU and Antoneta BRATCU

"Dunarea de Jos" University of Galati,
Department of Control Systems and Electronics,
111, Domneasca Street, 6200, Galati, Romania

Abstract: This paper deals with the modelling of a particular class of manufacturing lines, governed by a decentralised control strategy so that they balance themselves. Such lines are known as "*bucket brigades*" and also as "*TSS lines*", after their first implementation, at Toyota, in the 70's. A first study of their behaviour was based upon modelling as *stochastic dynamic systems*, which emphasised, in the frame of the so-called "Normative Model", a sufficient condition for self-balancing, that means for autonomous functioning at a steady production rate (stationary behaviour). Under some particular conditions, a simulation analysis of TSS lines could be made on *non-linear block diagrams*, showing that the state trajectories are piecewise continuous in between occurrences of certain discrete events, which determine their discontinuity. TSS lines may therefore be modelled as *hybrid dynamic systems*, more specific, with autonomous switching and autonomous impulses (jumps). A stability analysis of such manufacturing lines is allowed by modelling them as hybrid dynamic systems with discontinuous motions.

Keywords: production systems, self-organising systems, hybrid dynamic systems, convergence analysis.

1. INTRODUCTION

The idea of bucket brigades was first implemented in the 70's, in order to increase flexibility of different production lines, especially when products have extreme seasonalities or short life-cycles. The Toyota Sewn Management System (TSS), registered trademark of Aisin Seiki Co. Ltd., a subsidiary of Toyota, was the first case putting into practice a way of organising workers on a flow line so that the line balances itself. Workers, fewer than workstations, are allowed to walk to adjacent workstations to continue work on an item, each of them independently following a simple rule – the *TSS Rule* – that determines what to do next.

TSS lines have offered a new way of organising work on a production line, as an alternative to the traditional point of view, such as classical assembly line, where the station with the greatest work content determines the production rate. Opposite to the strict assignment of equipment and tasks to workstations, the well-known concept of "*workstation*" is given a new meaning, as the equipment are, in fact, human operators, which are not strictly assigned to certain workstations, but can move among them.

Flow manufacturing lines can be found wherever "products" may be imagined to move along, from worker to worker, for example, as in an assembly line: products are progressively assembled as they move down the line toward completion. When following the TSS Rule, a flow line is spontaneously maximally productive and autonomously maintains its optimal production rate, without any conscious intervention. The self-balancing emerges like an intrinsic property of the line, avoiding a solution based on the assembly line balancing (ALB) classical problem. As it is known, this problem is NP-complete. The cycle time of the line – the most used optimisation criterion in ALB approach – is in this case implicitly minimised.

An instance of the product is called an item. Each worker has an index, as higher as he is closer to the end of the line. A workstation can process at most one item at a time, requiring precisely one worker to perform the processing. Workers move according to the *TSS Rule*:

Forward part – Remain devoted to a single item and process it on successive workstations (where at any station the worker of higher index has priority). If your item is taken by your successor (or if you are the last worker and you complete processing the item), then relinquish the item and begin to follow the *backward part*.
Backward part – Walk back and take over the item of your predecessor (or, if you are the first worker, pick up raw materials to start a new item). Begin to follow the *forward part*.

Two aspects may be noted about the TSS Rule: (1) a worker can be blocked during the forward phase, if trying to enter an occupied station, and (2) during the backward phase, each worker must, in fact, interrupt his predecessor and take over his work.

Different approaches of particular flexible lines, quite close to the logic of a TSS line, can be encountered in the literature. It is shown that TSS lines regarded as dynamic systems can have very complicated and even chaotic behaviour (Yoshida, *et al.*, 1983), and that is why they were primarily treated under some simplifying but quite realistic assumptions. The implications of using human operators – such as motivation, mentality, responsibility of workers – have been ignored in a first modelling approach, considering that this would pointlessly complicate the model. Identifying workers with workstations, some authors propose that line balancing be achieved by clever management and work-in-process inventory (Ostolaza, *et. al*, 1990), whereas some simulation studies use a simple model of workers, where all are identical and proceed at a single common velocity across all stations. Having also assumed that the processing time at each station is normally distributed about its mean, a linear deterministic average behaviour of the line is obtained, emphasising the convergence of state trajectories – as defined below – to a fixed point (Schroer, *et al*, 1991).

Nevertheless, it is more realistic to consider each worker as a working velocity function depending on his position on the line. Based upon this assumption – and on other assumptions, forming together “The Normative Model” – the modelling of TSS lines as *stochastic dynamic systems* has allowed to establish a sufficient condition for obtaining a stationary behaviour, that is a steady production rate (Bartholdi and Eisenstein, 1996a). Thus, it has been proved that a bucket brigade production line is spontaneously maximally productive if workers are sequenced from slowest to fastest. This yields a stable partition of work, corresponding to a fixed point in the system’s state space. The same authors studied some practical implications of bucket brigades (Bartholdi and Eisenstein, 1996b), as well as all possible asymptotic behaviour of lines with two or three workers, each characterised by a constant work velocity (Bartholdi, *et al.*, 1999c).

In a recent work, simulations carried out on *non-linear block diagrams* have shown that the simplifying assumptions of The Normative Model induce the discontinuity of the state trajectories, due to occurrence of certain discrete events (Bratcu and Mînzu, 1999). Two *hybrid phenomena* were identified. Therefore, it has appeared necessary to reconsider the modelling approach in order to embed both the continuous behaviour and the impact of discrete events, that is to regard the TSS lines as *hybrid dynamic systems*. According to the taxonomy of Branicky, *et al.* (1998), such manufacturing lines are *hybrid dynamic systems with autonomous switching and autonomous jumps* (see also Flaus, 1998). Using concepts within a unitary *hybrid model with discontinuous motions* (Ye, *et al.*, 1995), the existence of the stationary behaviour – in particular, of the self-balancing – may be related to the stability analysis of hybrid dynamic systems.

This paper first presents the basic assumptions for modelling TSS lines. Then it will be summarised the main approach from the literature: the modelling as stochastic dynamic systems. A non-linear model of a TSS line will be presented next, leading to the necessity of modelling in the frame of the hybrid dynamic systems theory. Finally, a conclusion will be listed.

2. BASIC MODELLING ASSUMPTIONS

TSS lines are self-organising, therefore the need for centralised planning and management is reduced. Their operation is simple: each worker carries an item towards completion; when the last worker finishes his item, he sends it off and then walks back upstream to take over the work of his predecessor, who walks back and takes over the work of his predecessor and so on until, after relinquishing his item, the first worker walks back to the start to begin a new item. A worker might catch up to his successor and be blocked from proceeding. The TSS Rule requires that the blocked worker remain idle until the station is available. Only the last worker is never blocked and he determines the line productivity.

Let m be the number of workstations and n be the number of workers. The Normative Model is the simplest model of the dynamics of TSS lines, based on the following assumptions (Bartholdi and Eisenstein, 1996a):
a) *insignificant walk-back time*: the total time to finish a product is much greater than the total time for the workers to handoff their work and walk back to get more work;
b) *total ordering of workers by velocity*: each worker i is modelled by a velocity function, $v_i(x)$, giving his instantaneous work velocity at position $x \in [0;1]$;
c) *smoothness and predictability of work*: the standard work content required by an item is spread continuously and uniformly along the flow line, whose length is normalised to 1 and partitioned into intervals corresponding to workstations (see figure 1).

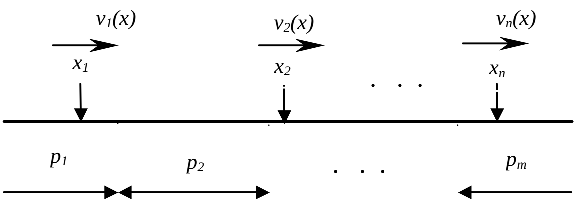

Fig. 1. The standard work content split into workstations and the workers positions ($\sum_{j=1}^{m} p_j = 1$).

x_i gives the position of the i-th worker and represents the cumulative fraction of work content completed on his item at a given moment. The vector of workers positions $\underline{x} = (x_1, x_2, ...x_n)$ represents the state of the system at any time. An *iteration* is the time elapsed between two successive handoff moments, which are also called *reset* moments.

The attention in the state space is restricted to the sequence $\{\underline{x}^{(0)}, \underline{x}^{(1)}, \underline{x}^{(2)}, ... \underline{x}^{(k)}, ...\}$ of positions immediately after reset (where $x_1^{(k)}=0$, $x_{n+1}^{(k)}=1$). There is no need for now to focus on the line evolution within an iteration, that means between the reset moments. The sequence above is called the *orbit* beginning at $\underline{x}^{(0)}$.

Let f be the function that maps the vector of workers reset positions so that $\underline{x}^{(k+1)}=f(\underline{x}^{(k)})$. The orbits $\{\underline{x}^{(k+1)}=f^k(\underline{x}^{(0)})\}_{k=0,1,...}$, determined by the initial conditions $\underline{x}^{(0)}$, describe the line behaviour. The *stationary behaviour* is defined by the *fixed points* of f, which are points of balancing.

3. TSS LINES AS STOCHASTIC DYNAMIC SYSTEMS

Hereafter is presented the main modelling and analysis approach of TSS lines from the literature. As it was said above, proofs have been made in a stochastic context; they have been skipped below (see Bartholdi and Eisenstein, 1996a).

The first result states the *existence* of at least one fixed point, $\underline{x}^*=f(\underline{x}^*)$, for any TSS line, meaning that there exist worker positions $\underline{x}^*$ such that, if workers start at positions $\underline{x}^*$, then they will always reset to $\underline{x}^*$. Therefore, balancing is always at least theoretically possible.

It is said that worker j is faster than worker i – written as $v_i \prec v_j$ – if

$$\sup_{x\in[0,1]} \left(\frac{v_i(x)}{v_j(x)} \right) < 1,$$

meaning that j is faster than i for any operation from the line. The above notation helps at stating a sufficient condition for the *uniqueness* of a fixed point of a TSS line, that is workers being sequenced *from slowest to fastest* (well order of workers). If this condition is not fulfilled, then the line may have multiple fixed points. Keeping the well order of workers, it has been proved that any orbit of worker positions, $\{x^{(t+1)}=f^t(x^{(0)})\}$, converges to the unique fixed point.

As it is known, the *production rate* of a production line may be defined as the number of items completed in a time unit, as well as the time for processing an item. If there exists a stationary regime, the second definition corresponds to the *cycle time* of the line.

Let $t_i(x,x')$ be the time required to worker i, if not blocked, for walking from position x to position x', $0 \le x \le x' \le 1$:

$$t_i(x,x') = \int_x^{x'} \frac{dz}{v_i(z)}$$

Let P_k be the cumulative amount of work performed on an item when it has just left the station k:

$$P_0=0, \quad P_k = \sum_{j=0}^{k} p_j, \quad k=1,2,...m \tag{1}$$

According to (1), the interval $(P_{k-1};P_k)$ represents the work content assigned to station k. Because it is claimed that only one worker can use a certain station at a given time, no two xi's can assume values within the same interval $(P_{k-1};P_k)$. There are defined:

$$\underline{x} = P_{k-1}, \text{ if } x \in [P_{k-1};P_k)$$
$$\overline{x} = P_k, \quad \text{if } x \in (P_{k-1};P_k]$$

The interval $[x_i^{(t)};x_{i+1}^{(t)}]$ may be regarded like a dynamic partition of the work content assigned to worker i during iteration t. Let $a_i^{(t)}$ be the time that would take to worker i to complete his suggested share of work, including both effective work time and possible delays because of blocking. This time is called *allocation*. Thus, one can write:

$$\begin{cases} a_n^{(t)} = t_n(x_n^{(t)},1) \\ a_i^{(t)} = t_i(x_i^{(t)},x_{i+1}^{(t)}) + \\ \qquad + max\left\{0, t_{i+1}(x_{i+1}^{(t)},\overline{x_{i+1}^{(t)}}) - t_i(x_i^{(t)},\underline{x_{i+1}^{(t)}})\right\}, \\ \qquad i=1,2,...n-1 \end{cases}$$

Pure work time allocations are called *simple*, different from *delayed* allocations, which include also delays due to blocking. They are respectively expressed by the two relations below:

$$a_i^{(t)} = t_i(x_i^{(t)},x_{i+1}^{(t)}),\ \ i=1,2,...n$$
$$a_i^{(t)} = t_i(\underline{x_{i+1}^{(t)}},x_{i+1}^{(t)}) + t_{i+1}(x_{i+1}^{(t)},\overline{x_{i+1}^{(t)}}),\ \ i=1,2,...n$$

The main result states that, if workers' velocities are constant, with $v_1<v_2<...<v_n$, and if workers are never blocked, then the line converges *exponentially fast* to the unique fixed point:

$$\underline{x}^* = \begin{bmatrix} 0 \\ \frac{v_1}{\sum_{j=1}^{n} v_j} \\ ... \\ \frac{\sum_{j=1}^{i-1} v_j}{\sum_{j=1}^{n} v_j} \\ ... \\ \frac{\sum_{j=1}^{n-1} v_j}{\sum_{j=1}^{n} v_j} \end{bmatrix} \tag{2}$$

where the production rate is the largest possible: $\sum_{j=1}^{n} v_j$. The line behaviour is linear, being described by the following equations:

$$\begin{cases} a_1^{(t+1)} = a_n^{(t)} \\ a_i^{(t+1)} = \frac{v_{i-1}}{v_i} \cdot a_{i-1}^{(t)} + (1 - \frac{v_{i-1}}{v_i}) \cdot a_n^{(t)},\ \ i=1,2,...n \end{cases},$$

representing a *linear dynamic system*.

By rewriting these equations in the form:

$$a^{(t+1)} = T\,a^{(t)} \tag{3}$$

where T denotes the *transition matrix* of a *finite state Markov chain* that is irreducible and aperiodic (Resnick, 1992), one can note that the convergence of iterates $\{a_i^{(t)}\}_{t=0}^{\infty}$ is guaranteed for any $i=1,2,...n$, yielding the convergence of the TSS line orbit $\{x^{(t)}\}_{t=0}^{\infty}$. It is said that the line has a *stationary behaviour*. Note also that, if workers are sequenced from slowest to fastest, then the largest allocation, which is the cycle time of the line, converges from above and it is guaranteed to have decreased after each completion of n items. The production rate improvement is possible to a limit not depending on the starting positions of the workers. Yet, one can notice that the well order is only a sufficient condition for the stationary behaviour, since there are cases of other than slowest-to-fastest sequences of workers when the line converges to a fixed point not depending on the workers' initial positions (Bratcu and Mînzu, 1999).

4. A NON-LINEAR MODEL OF TSS LINES

Different simulation cases referred in the literature have shown typical non-linear dynamics – such as limit cycles or behaviour depending of the initial positions of workers, or either on the partition of work among workstations – occurring when the "proper" sequence is not respected or/and it happens that workers be blocked (the so-called *complicated behaviour*). To understand it, the simulation study of state trajectories has been extended in between the reset moments. Thus, a non-linear model of TSS lines was built for simulation purpose, in the particular case of constant work velocities, but it can be easily extended in the general case (Bratcu and Mînzu, 1999). As the matter of fact, as it is shown next, the hybrid dynamics of such lines is already reflected in this model.

First, notice that the system exhibits *piecewise linear dynamics*, since any worker on the line can have only *two states* during an iteration: either he moves with a certain nonzero velocity, or he is blocked by the next worker when trying to continue work on the next station (velocity becomes zero). Two *hybrid phenomena* (Flaus, 1998) may be identified. The first one, called *autonomous switching*, appears at the transition between the two states above mentioned. The state trajectory remains continuous in this case. It exhibits discontinuities only in the reset moments, when the last worker reaches the line end. The line reset may thus be modelled as the second hybrid phenomenon, called *autonomous jump*. The jump phenomenon results from the simplifying assumption that workers move back infinitely fast, whereas they practically cannot do. Thus, it is ensured a *single moment of reset* for all workers.

The occurrence of the jump could be avoided by considering that workers move back with a single common velocity, much greater that each individual work velocity. Nevertheless, the advantage of having a continuous state trajectory would be covered by a greater complexity of the model, since in that case workers would reset at *different* moments.

Notation:
$tr_1, tr_2, \ldots tr_p, \ldots$ = the sequence of successive moments of reset;
tr_p^{-0} = the moment immediately *before* the *p*-th reset;
tr_p^{+0} = the moment immediately *after* the *p*-th reset;

$$sign(y(t)) = \begin{cases} 1, & y(t) > 0 \\ -1, & y(t) < 0 \end{cases}$$

The evolution between any two consecutive moments of reset is characterised by: $x_n^{(p)}(t)<1$, $\forall\ t \in [tr_p, tr_{(p+1)})$, $\forall\ p \in \mathbf{N}^*$.

$$If\ x_n(t)<1, \begin{cases} \dot{x}_1(t) = \dfrac{v_1}{2} + \dfrac{v_1}{2}\cdot sign\left\{\dfrac{1}{2}\cdot\sum_{k=1}^{m}\left\{p_k\cdot\left(sign\left(x_2(t)-P_{k-1}\right) - sign\left(x_1(t)-P_{k-1}\right)\right)\right\}\right\} \\ \dot{x}_2(t) = \dfrac{v_2}{2} + \dfrac{v_2}{2}\cdot sign\left\{\dfrac{1}{2}\cdot\sum_{k=1}^{m}\left\{p_k\cdot\left(sign\left(x_3(t)-P_{k-1}\right) - sign\left(x_2(t)-P_{k-1}\right)\right)\right\}\right\} \\ \ldots \\ \dot{x}_{n-1}(t) = \dfrac{v_{n-1}}{2} + \dfrac{v_{n-1}}{2}\cdot sign\left\{\dfrac{1}{2}\cdot\sum_{k=1}^{m}\left\{p_k\cdot\left(sign\left(x_n(t)-P_{k-1}\right) - sign\left(x_{n-1}(t)-P_{k-1}\right)\right)\right\}\right\} \\ \dot{x}_n(t) = v_n \end{cases} \quad (4)$$

$$otherwise(\ jump: x_n(t)=1 \Leftrightarrow t \equiv t_{rp}\ ), \begin{cases} x_1(t_{rp}) = 0 \\ x_2(t_{rp}) = x_1(t_{rp}^{-0}) \\ x_3(t_{rp}) = x_2(t_{rp}^{-0}) \\ \ldots \\ x_n(t_{rp}) = x_{n-1}(t_{rp}^{-0}) \end{cases}, \qquad p=1,2,3,\ldots$$

The non-linear model of a TSS line is given in (4), with the initial condition $\underline{x}^{(0)}=\underline{x}(0)=x_0=[x_{10}\ x_{20}\ \ldots\ x_{n0}]$. This model was used to build a non-linear block Simulink diagram for simulation, where workers were modelled by *reset integrators*.

5. TSS LINES AS HYBRID DYNAMIC SYSTEMS

This section shows that the modelling tools provided by the theory of hybrid dynamic systems are suitable to approach TSS manufacturing lines. Involving both continuous-valued and discrete-valued variables, the evolution of a hybrid dynamic system is given by equations of motion that generally depend on both. The continuous and discrete dynamics interact when the continuous state hits certain sets in the continuous state space. In view of the taxonomy proposed by Branicky, *et al.* (1998), TSS lines can be regarded as *hybrid systems with autonomous switching and autonomous jumps*.

The *autonomous switching* is modelled as follows:

$$\begin{cases} \dot{\underline{x}}(t) = h\left(\underline{x}(t), \underline{q}_w(t)\right) \\ \underline{q}_w^+(t) = n\left(\underline{x}(t), \underline{q}_w(t), \underline{q}_s(t)\right) \\ \underline{q}_s^+(t) = m\left(\underline{x}(t), \underline{q}_s(t)\right) \end{cases} \quad (5)$$

where: $\dot{\underline{x}}(t)=[x_1(t)\,x_2(t)\ldots x_n(t)]^T \in [0;1]^n$ is the continuous state space vector (the instantaneous positions of workers), $\underline{q}_w(t) \in \{0,1\}^{n-1}\times\{1\}$ denotes the discrete states of workers (*1*-working, *0*-blocked,), $\underline{q}_s(t) \in \{0,1\}^m$ represents the discrete states of workstations (*1*-occupied, *0*-available). Notation $(\times)^+$ denotes the following (discrete) state.

Let $\underline{v} = [v_1\ v_2 \ldots v_n]^T$ be the vector of work velocities. Function *h* is given by:

$$\dot{\underline{x}}(t) = \underbrace{\begin{bmatrix} q_{w_1}(t) & & & 0 \\ & q_{w_2}(t) & & \\ & & \ldots & \\ 0 & & & q_{w_n}(t) \end{bmatrix}}_{A_q = diag\{\underline{q}_w(t)\}} \cdot \underline{v} \quad (6)$$

Remembering that notation (1) $P_j = \sum_{k=0}^{j} p_k,\ j=\overline{1,m},\ P_0 = 0,\ P_m = 1$ denotes the cumulative work content on an item immediately after it leaves the *j*-th station (see section 2), functions *n* and *m* may be respectively expressed by (7) and (8):

$$q_{w_i}^{+}(t)=\begin{cases}0, q_{w_i}(t)=1, \exists j:(x_i(t)=P_{j-1})\wedge(q_{s_j}=1)\\ 1, q_{w_i}(t)=0, \exists j:(x_i(t)=P_{j-1})\wedge(q_{s_j}=0),\\ q_{w_i}(t), otherwise\end{cases}$$
$$i=\overline{1,n-1} \quad (7)$$
$$q_{w_n}^{+}(t)=1$$

$$q_{s_j}^{+}(t)=\begin{cases}0, q_{s_j}(t)=1, \exists i: x_i(t)=P_j\\ 1, q_{s_i}(t)=0, \exists i: x_i(t)=P_{j-1},\\ q_{s_j}(t), otherwise\end{cases}$$
$$j=\overline{1,m} \quad (8)$$
$$q_{w_n}^{+}(t)=1$$

The *autonomous jumps* may be modelled by:

$$\begin{cases}\dot{\underline{x}}(t)=h(\underline{x}(t),\underline{q}_w(t)), & \underline{x}(t)\notin M\\ \underline{x}^{+}(t)=\mathrm{J}(\underline{x}(t)), & \underline{x}(t)\in M\end{cases} \quad (9)$$

where $M=\{m_1, m_2, \ldots, m_{n-1}, 1\} \in [0;1]^{n-1}\times\{1\}$ (jumps describe the handoff; they take place when the line resets, that means when $x_n(t)=1$). Function h is given by (6) and function J is detailed in:

$$\mathrm{J}(\underline{x}(t))=\underbrace{\begin{bmatrix}0 & 0 & \ldots & 0 & 0\\ 1 & 0 & 0 & \ldots & 0\\ 0 & 1 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & 0 & 0\\ 0 & \ldots & 0 & 1 & 0\end{bmatrix}}_{P}\cdot\underline{x}(t) \quad (10)$$

The model of a TSS line viewed as a hybrid dynamic system with autonomous switching and autonomous jumps is obtained by simply coupling the two models above (relations (5) and (9)):

$$\begin{cases}\dot{\underline{x}}(t)=h(\underline{x}(t),\underline{q}_w(t))=A_q\cdot\underline{v},\\ \qquad \underline{x}(t)\notin M \quad (x_n(t)\neq 1)\\ \underline{x}^{+}(t)=\mathrm{J}(\underline{x}(t))=P\cdot\underline{x}(t),\\ \qquad \underline{x}(t)\in M \quad (x_n(t)=1)\\ \underline{q}_w^{+}(t)=n(\underline{x}(t),\underline{q}_w(t),\underline{q}_s(t))\\ \underline{q}_s^{+}(t)=m(\underline{x}(t),\underline{q}_s(t))\end{cases} \quad (11)$$

with the initial condition $0\le x_{1_0}<x_{2_0}<\ldots<x_{n_0}<1$.

An unitary approach of hybrid dynamic systems, quite close to that of the classical dynamic systems theory and more suitable to formulate the stability analysis problem, was proposed by Ye, *et al.* (1995). Well-known concepts, such as time space, motion, invariant set and equilibrium, are extended in order to embed special features of hybrid dynamic systems. It is shown below how some of these results may be applied in the case of TSS lines.

A TSS manufacturing system may be described as a 5-uple $\{T, X, A, S, T_0\}$ – which is a hybrid dynamic system (see Ye, *et al.*, 1995) – as follows:

$T=\mathbf{R}^{+}$ (the system evolves continuously in time);
$T_0=\{t_0 | t_0\in T\}$ (the set of *initial moments*);
$X=\mathbf{R}^{n}$ (the set of *states*);
$X\supset A=\{a | a=[x_{1_0}\ x_{2_0}\ldots x_{n_0}]^T\}$ (the set of *initial states*);
$S\subset\{p(t,a,t_0)=\underline{x}(t,a,t_0)=[x_1(t,a,t_0)\ x_2(t,a,t_0)\ldots x_n(t,a,t_0)]^{T}\}$,
where $p(t_0,a,t_0)=a$, are the *motions* (trajectories) of the system.

One may observe that, if all workers are of constant velocity along the line, when being sequenced from slowest to fastest ($v_1<v_2<\ldots<v_n$ – assumption of well ordering) and never blocked, motions are implicitly given as the solutions of the differential equation:

$$\dot{\underline{x}}(t,a,t_0)=[v_1\ v_2\ldots\ v_n]^T \quad (12)$$

with the initial condition $0\le x_{1_0}<x_{2_0}<\ldots<x_{n_0}<1$.

Definition 1 - invariant set (Ye, *et al.*, 1995):
Let $\{T, X, A, S, T_0\}$ be a hybrid dynamic system. A set $M\subset A$ is called invariant of system S if

$$\forall t\in T_{a,t_0}, \forall t_0\in T_0, \forall p(\cdot,a,t_0)\in S:$$
$$a\in M\Rightarrow p(t,a,t_0)\in M$$

It is said that M is an invariant of S or (S, M) is invariant.

Definition 2 - equilibrium (Ye, *et al.*, 1995):
$x_0\in A$ is called an equilibrium of a hybrid dynamic system $\{T, X, A, S, T_0\}$ if the set $\{x_0\}$ is invariant with respect to S.

In other words, any motion starting from a state of an invariant remains within that invariant. An equilibrium is a state that, once it is reached, it will be never quitted by any motion of the system. One may note that the functioning rules of TSS line – mainly the rule of handoff – allows formulating the following

Proposition:
If the system S, described by relation (12), is reinitialised every time when $x_n(t)=1$ according to

$$\begin{cases}x_1(t)=0\\ x_i(t)=x_{i-1}(t^{-}),\ i=\overline{2,n}\end{cases} \quad (13)$$

then the subset $[0;1]^n$ of the state space is an invariant of the system *S*.
Proof:

Let *a* be an initial state from $[0;1]^n$ and t_0 be an initial moment. One may consider $t_0=0$ without loss of generality. It must be proved that all motions described by $\underline{x}(t,a,t_0)$ remain in $[0;1]^n$.

Let r_1 be the first moment when $x_n(t)=1$, that is the end of the first iteration or, equivalently, the first reset moment. Taking into account notations from section 2, one may write:

$$\underline{x}(0,a,0) = a = [\, x_{1_0} \ x_{2_0} \ \dots x_{n_0} \,]^T = \underline{x}^{(0)} \in [\,0;1\,]^n$$
$$\underline{x}(r_1,a,0) = [\, x_1(r_1) \ x_2(r_1) \ \dots \ x_n(r_1)\,]^T = \underline{x}^{(1)}$$

From now on, notations for initial state and initial moment are dropped. Let $\underline{y}^{(0)} = \underline{x}(r_1^-)$. From (13) one may write:

$$\begin{cases} x_1(r_1) = 0 \\ x_i(r_1) = x_{i-1}(r_1^-) \Leftrightarrow \underline{x}^{(1)} = P \cdot \underline{y}^{(0)} \\ \qquad i = \overline{2,n} \end{cases} \qquad (14)$$

where matrix *P* models reset jumps (previously given in (10)). From (12) and from the assumption of well ordering it follows that:

$$\begin{cases} y_i^{(0)} = x_i^{(0)} + v_i \cdot r_1 \\ \qquad < x_{i+1}^{(0)} + v_{i+1} \cdot r_1 = y_{i+1}^{(0)}, \\ \qquad i = \overline{1,n-1} \\ y_n^{(0)} = 1 \end{cases}$$

So, $x_1^{(0)} \le y_1^{(0)} < y_2^{(0)} < \dots < y_{n-1}^{(0)} < y_n^{(0)} = 1$, which yields that $\underline{y}^{(0)} \in [0;1]^n$. Obviously, the positions of workers within the first iteration respect $\underline{x}^{(0)} \le \underline{x}(t) < \underline{y}^{(0)}$. It follows that $\underline{x}(t) \in [0;1]^n, \forall t \in [0;r_1)$ and, using (14), that $\underline{x}^{(1)}$ Î $[0;1]^n$.

The same reasoning may be used in order to prove by induction after *k* that $\underline{x}(t) \in [0;1]^n, \forall t \in [r_{k-1};r_k),\ k = 1,2,\dots$ Note that $\underline{x}^{(k)}$ may be regarded as the initial state of the *k*-th iteration. The goal follows. ð

When modelling a TSS manufacturing line as a hybrid dynamic system, one may superficially think that the balancing point of the line – expressed by relation (2) – might be characterised as an equilibrium of the system, in view of definition 2. This is not true, because, in fact, the motions do not converge to a single point of the state space (this would mean that workers be stopped), but to *a pattern* of moving, where each worker *i* repeats the execution of a work content in the interval $\left[\frac{\sum_{j=1}^{i-1} v_j}{\sum_{j=1}^{n} v_j} ; \frac{\sum_{j=1}^{i} v_j}{\sum_{j=1}^{n} v_j} \right]$. The stationary behaviour is a *periodical* one.

6. CONCLUSION AND FURTHER DEVELOPMENT

Being self-organising systems, able to autonomously reach a periodical stationary behaviour, expressed by the optimal production rate, TSS lines – called also bucket brigades – have been primarily treated as stochastic dynamic systems.

This paper has presented a new modelling approach of TSS manufacturing lines, showing that they may be modelled first as non-linear dynamic systems – for simulation purpose – and, even more correctly, as hybrid dynamic systems, namely with autonomous switching and autonomous jumps. Within this latter frame, another modelling approach, based upon emphasising discontinuous motions, has opened a view to the stability analysis. An invariant set of the system – in terms of hybrid dynamic systems theory of stability – has been deduced.

Remember that the results obtained in modelling and analysis of TSS lines have been based upon some simplifying, even quite realistic assumptions (The Normative Model), whose main weakness was to model workers as simple work velocities. TSS lines behaviour would become very complicate if taking into account the psychological aspects induced by using human operators. An interesting further development of the work presented herein would be to try treating the general case, and this is probably achievable in the frame of chaos theory. On the other hand, the actual trend of implementing TSS strategy on flow lines is oriented to using robots, obviously more easily to control. In this context, the Normative Model becomes even more suitable for analysis purpose and the TSS lines average behaviour is much closer to linear deterministic.

REFERENCES

Bartholdi, J.J. and D.D. Eisenstein (1996a). A production line that balances itself. *Operations Research*, **Volume No 44**(1), pp. 21-34.
Bartholdi, J.J. and D.D. Eisenstein (1996b). The agility of bucket brigade production lines. In: *Proceedings of the Conference on Flexible and Intelligent Manufacturing*.
Bartholdi, J.J., L.A. Bunimovich and D.D. Eisenstein (1999c). Dynamics of 2- and 3-

worker bucket brigade production lines. *Operations Research*, **47**(3), pp. 488-491.

Branicky, M.S., V.S. Borkar and S.K. Mitter (1998). A Unified Framework for Hybrid Control: Model and Optimal Control Theory. *IEEE Transactions on Automatic Control*, **Volume 43, No 1,** pp. 31-45.

Bratcu, A. and V. Mînzu (1999). A Simulation Study of Self-balancing Production Lines. In: *Proceedings of the 1999 IEEE International Conference on Intelligent Engineering Systems - INES '99*, pp. 165-170, Stará Lesná, Slovakia.

Flaus, J.M. (1998). Modeling and analysis of hybrid dynamical systems: an introduction. *European Journal of Automation Systems*, **Volume 32, No 7-8**, pp. 797-830.

Ostolaza, J., J. McClain and J. Thomas. (1990). The use of dynamic (state-dependent) assembly line balancing to improve throughput. *J. Mfg. Oper. Mgt.*, **3**, pp. 105-133.

Resnick, S.I. (1992). *Adventures in Stochastic Processes*, Birkhäuser, Boston.

Schroer, B.J., J. Wang and M.C. Ziemke (1991). A look at TSS through simulation. *Bobbin magazine*, July 1991, pp. 114-119.

Ye, H., A.N. Michel and L. Hou (1995). Stability Theory for Hybrid Dynamical Systems. In: *Proceedings of the 34th Conference on Decision and Control*, pp. 2679-2684, New Orleans, U.S.A.

Yoshida, T., H. Mori and H. Shigematsu. (1983). Analytic study of chaos of the tent map: band structures, power spectra and critical behaviours. *Journal of Statistical Physics*, **31**(2), pp. 279-308.